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LETTER TO THE EDITOR

The sine-Gordon equation and the trace method

Zheng Weiming

Shanghai Institute of Computer Technology, Shanghai, People's Republic of China

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Abstract. The trace method which has been proposed by Wadati and Sawada is applied to the sine-Gordon equation. The N soliton solution is derived.

The trace method which has been applied to the KdV [1], MKdV [2], KP [3] and some other equations [4] is useful for understanding these equations [1]. The N soliton solution and some other results of these equations have been derived through the trace method.

The present letter deals with an application of the trace method to the sine-Gordon equation as follows:

$$U_{xt} + \sin U = 0 \qquad U(x, t) \to \text{constant as } x \to \infty. \tag{1}$$

Wadati and Sawada have pointed out that it is very difficult to solve the sine-Gordon equation by means of the trace method directly [2]. In order to avoid dealing with the transcendental non-linearity sin U, it is convenient to re-express the equation in terms of $V = U_x$, $W = \cos U$, i.e.

$$V_{xt} = -WV \qquad W_x = V_t V. \tag{2}$$

Substituting the formal series

$$\begin{aligned} V &= V^{(1)} + V^{(3)} + \ldots + V^{(2n+1)} + \ldots \\ W &= 1 + W^{(2)} + V^{(4)} + \ldots + W^{(2n)} + \ldots \end{aligned} \tag{3}$$

into equations (2), we obtain a set of equations for $W^{(2n)}$ and $V^{(2n+1)}$ $(n = 0, 1, 2, \ldots)$:

$$W^{(0)} = 1 \qquad V^{(1)}_{xt} = -V^{(1)} \qquad W^{(2)}_x = V^{(1)}_t V^{(1)}$$

$$V^{(3)}_{xt} = -V^{(3)} - W^{(2)} V^{(1)}$$

$$\vdots$$

$$W^{(2n)}_x = \sum_{r=0}^{n-1} V^{(2r+1)}_t V^{(2n-2r-1)} \tag{4}$$

$$V^{(2n+1)}_{xt} = -\sum_{r=0}^{n} W^{(2r)} V^{(2n-2r+1)} \tag{5}$$

$$\vdots$$

We can solve the set of equations iteratively:

$$V^{(1)}=4\sum_{r=1}^{N}\Phi_r^2(x,t)$$

$$W^{(2)}=-\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{\Phi_m^2(x,t)\Phi_n^2(x,t)}{(k_m+k_n)}(k_m^{-1}+k_n^{-1})$$

$$V^{(3)}=-4\sum_{r=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{\Phi_r^2(x,t)\Phi_m^2(x,t)\Phi_n^2(x,t)}{(k_r+k_m)(k_m+k_n)}.$$

Here $\Phi_n=A_n(0)\exp(k_nx-t/4k_n)=\Phi_n(x,t)$, k_n are positive and distinct constants $A_n(0)$ are real or pure imaginary $(n=1,2,\ldots,N)$. We introduce an $N\times N$ matrix whose elements are given by $B_{mn}=\int_{-\infty}^{x}\Phi_m(y,t)\Phi_n(y,t)\,dy=[1/(k_m+k_n)]\Phi_m(x,t)\Phi_n(x,t)$.

With the matrix B, $V^{(1)}$, $W^{(2)}$, and $V^{(3)}$ are expressed as

$$V^{(1)}=4\,\mathrm{Tr}(B_x)\qquad W^{(2)}=2\mathrm{Tr}(B_{xt}^2)\qquad V^{(3)}=-4\mathrm{Tr}(B_xB^2).$$

In general, we can prove that

$$V^{(2n+1)}=4(-1)^n\,\mathrm{Tr}(B_xB^{2n})\tag{6}$$

$$W^{(2n)}=(2/n)(-1)^{n-1}\,\mathrm{Tr}(B_{xt}^{2n})\tag{7}$$

satisfy equations (4) and (5). First, it is easy to see that

$$\mathrm{Tr}[(B^{2n+1})_x]=(2n+1)\,\mathrm{Tr}(B_xB^{2n}).\tag{8}$$

Therefore

$$V^{(2n+1)}=[4/(2n+1)](-1)^n\,\mathrm{Tr}[(B^{2n+1})_x]$$

$$V_{xt}^{(2n+1)}=[4/(2n+1)](-1)^n[\mathrm{Tr}(B^{2n+1})]_{txx}.\tag{9}$$

Now we make the following definitions.

(i) σ is a cyclic transformation, $\sigma(1,2,\ldots,2n+1)=(2,3,\ldots,2n+1,1)$.

(ii) T is an operator for a subscript variable, defined by $T(F(k_{r_1},k_{r_2},\ldots,k_{r_{2n+1}}))=F(k_{r_{\sigma(1)}},k_{r_{\sigma(2)}},\ldots k_{r_{\sigma(2n+1)}})$.

(iii) S is an order symmetry operator defined by

$$S=T^1+T^2+\ldots T^{2n+1}.$$

Obviously, we have that

(1) S is a linear operator

(2) $ST\neq S$, $TS=S$ and

(3) if f is a function of the $(2n+1)$th subscript variable and $T(f)=f$, then $S(f)=(2n+1)f$ and $S(gf)=(S(g))f$, where g is a function of the subscript variable.

From (8), we have

$$W^{(2n)}=(2/n)(-1)^{n-1}\,\mathrm{Tr}(B_{xt}^{2n})=4(-1)^{n-1}[\mathrm{Tr}(B_xB^{2n-1})]_t.\tag{10}$$

In the following we calculate the right-hand side of equation (5) and simplify the expression by writing $1, 2, \ldots, 2n+1$ instead of $r_1, r_2, \ldots, r_{2n+1}$:

$$-\sum_{r=0}^{n} W^{(2r)} V^{(2n-2r+1)}$$

$$= 4(-1)^{n+1} \sum_{1}\sum_{2} \cdots \sum_{2n+1}$$

$$\times \left(1 + 2\sum_{r=1}^{n} (k_1^{-1} + k_2^{-1} + \ldots + k_{2r}^{-1})(k_{2r} + k_{2r+1})\right) \Phi_1^2 \Phi_2^2 \ldots \Phi_{2n+1}^2$$

$$\times [(k_1 + k_2)(k_2 + k_3) \ldots (k_{2n} + k_{2n-1})]^{-1}$$

$$= 94(-1)^{n+1} \sum_{1}\sum_{2} \cdots \sum_{2n+1}$$

$$\times \left(1 + 2\sum_{r=1}^{x} (k_1^{-1} + k_2^{-1} + \ldots + k_{2r}^{-1})(k_{2r} + k_{2r+1})\right)$$

$$\times (k_{2n+1} + k_1) \Phi_1^2 \Phi_2^2 \ldots \Phi_{2n+1}^2$$

$$\times [(2n+1)(k_1 + k_2)(k_2 + k_3) \ldots (k_{2n} + k_{2n+1})(k_{2n+1} + k_1)]^{-1}.$$

From the well known Abelian transformation, we have

$$\sum_{r=1}^{n} (k_1^{-1} + k_2^{-1} + \ldots + k_{2r}^{-1})(k_{2r} + k_{2r+1}) = \sum_{i=1}^{n} (k_{2i-1}^{-1} + k_{2i}^{-1})\left(\sum_{r=i}^{n} (k_{2r} + k_{2n+1})\right)$$

$$S\left(1 + 2\sum_{r=1}^{n} (k_1^{-1} + k_2^{-1} + \ldots + k_{2r}^{-1})(k_{2r} + k_{2r+1})\right)(k_{2n+1} + k_1)$$

$$= S(k_{2n+1} + k_1) + 2S(k_{2n+1} + k_1) \sum_{i=1}^{n} (k_{2i-1}^{-1} + k_{2i}^{-1})\left(\sum_{r=i}^{n} (k_{2r} + k_{2r+1})\right)$$

$$= 2S(k_1) + 2S \sum_{i=1}^{n} \left[T^{2(n-i)+3}(k_{2n+1} + k_1) k_{2i-1}^{-1}\left(\sum_{r=i}^{n} (k_{2r} + k_{2r+1})\right) + T^{2(n-i+1)}(k_{2n+1} + k_1) k_{2i}\left(\sum_{r=i}^{n} (k_{2r} + k_{2r+1})\right)\right]$$

$$= 2S\left(k_1 + k_1^{-1} \sum_{i=0}^{n-1}\sum_{r=1}^{n-i} [(k_{2r} + k_{2r+1})(k_{2n-2i+1} + k_{2n-2i+2}) + (k_{2r-1} + k_{2r})(k_{2n-2i} + k_{2n-2i+1})]\right)$$

$$= 2Sk_1\left(k_1 + \sum_{i=0}^{n-1}\sum_{r=2}^{2n-2i+1} k_r k_{2n-2i+1} + \sum_{i=1}^{n-1}\sum_{r=2}^{2n-2i+1} k_r k_{2n-2i+2} + \sum_{r=2}^{2n+1} k_r k_1 + \sum_{i=0}^{n-1}\sum_{r=1}^{2n-2i} k_r k_{2n-2i} + \sum_{i=0}^{n=1}\sum_{r=1}^{2n-2i} k_r k_{2n-2i+1}\right)$$

$$= 2Sk_1^{-1}\left(\sum_{i \leqslant j} k_i k_j + \sum_{i>j} k_i k_j\right)$$

$$= 2(k_1^{-1} + k_2^{-1} + \ldots + k_{2n+1}^{-1})(k_1 + k_2 + \ldots + k_{2n+1})^2.$$

Therefore, we arrive at

$$-\sum_{r=0}^{n} W^{(2r)} V^{(2n-2r+1)}$$
$$= [4/(2n+1)](-1)^n \operatorname{Tr}(B_{xxt}^{2n+1})$$
$$= V_{xt}^{(2n+1)}.$$

Next we shall show that (6) and (7) satisfy equation (4):

$$\sum_{r=0}^{n-1} V_t^{(2r+1)} V^{(2n-2r-1)}$$
$$= \tfrac{1}{2}\left(\sum_{r=0}^{n-1} V_t^{(2r+1)} V^{(2n-2r-1)} + \sum_{r=0}^{n-1} V^{(2r+1)} V_t^{(2n-2r-1)}\right)$$
$$= 8(-1)^{n-1} \sum_{r=0}^{n-1} \left(\sum_1 \sum_2 \ldots \sum_{2n} [(k_1+k_2)\ldots(k_{2r}+k_{2r+1}) \times (k_{2r+2}+k_{2r+3})\ldots(k_{2n-1}+k_{2n})]^{-1} \Phi_1^2 \Phi_2^2 \ldots \Phi_{2n}^2\right)_t$$
$$= 8(-1)^{n-1}\left(\sum_1 \sum_2 \ldots \sum_{2n} \sum_{r=0}^{n-1} (k_{2r+1}+k_{2r+2})[(k_1+k_2)(k_2+k_3)\ldots \times (k_{2n-1}+k_{2n})]^{-1} \Phi_1^2 \Phi_2^2 \ldots \Phi_{2n}^2\right)_t$$
$$= 4(-1)^{n-1}[\operatorname{Tr}(B_x B^{2n-1})]_{xt}$$
$$= W_x^{(2n)}.$$

We have proved that (6) and (7) satisfy equations (4) and (5). Therefore we obtain the solution for equations (2) in the following form:

$$V = 4\operatorname{Tr}(B_x - B_x B^2 + B_x B^4 + \ldots) \neq 4\operatorname{Tr}[B_x(1+B^2)^{-1}]$$
$$= 4[\operatorname{Tr}(\tan^{-1} B)]_x = \frac{2}{\mathrm{i}}\left[\operatorname{Tr}\log\left(\frac{I+\mathrm{i}B}{I-\mathrm{i}B}\right)\right]_x$$

$$W = 1 + 2\{\operatorname{Tr}[B^2 - \tfrac{1}{2}B^4 + \ldots + (-1)^{n-1} n^{-1} B^{2n} + \ldots]\}_{xt} = 1 + 2[\operatorname{Tr}\log(1+B^2)]_{xt}.$$

From $V = U_x$, we have

$$U = 4\operatorname{Tr}(\tan^{-1} B) = \frac{2}{\mathrm{i}}\operatorname{Tr}\log\left(\frac{I+\mathrm{i}B}{I-\mathrm{i}B}\right). \tag{11}$$

We have to show that the above U satisfy equation (1). Due to $V = U_x$, we have

$$W_x = V_t V = U_{xt} U_x \tag{12}$$
$$W_{xx} = U_{xxt} U_x + U_{xt} U_{xx}.$$

Using $V_{xt} = -WV$, i.e. $U_{xxt} = -WU_x$ we obtain

$$W_{xx} = -WU_x^2 + U_{xt} U_{xx}.$$

Therefore we have

$$O = W_{xx}/U_x^2 + W - U_{xt}U_{xx}/U_x^2 = W_{xx}/U_x^2 - W_xU_{xx}/U_x^3 + W$$

$$= (W_x/U_x)_x \frac{dX}{dU} + W = \frac{d^2W}{dU^2} + W.$$

We solve $W_{uu} + W = 0$ and obtain

$$W = C_1 \cos U + C_2 \sin U \qquad C_1 \text{ and } C_2 \text{ constant.}$$

From (11) we see that U is an odd function of B, but W is an even function of B. Hence W is an even function of U and $C_2 = 0$. If we let $U = 0$, then from (11) $B = 0$, and $C_1 = C_1 \cos 0 = 1 = W$ (as $B = 0$). Therefore we arrive at $W = \cos U$. Using $W_x = V_tV$, we have $(\cos U)_x = U_{xt}U_x = (-\sin U)U_x$, i.e. $U_{xt} + \sin U = 0$. In conclusion, the U defined by (11) satisfy equation (1).

We can easily derive the Gelfand-Levitan integral equation and the eigenvalue problem associated with the sine-Gordon equation in a method similar to the MKdV equation, by replacing U by $V/2$ [2].

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